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$1/f$ noise in the Bak-Sneppen model

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We calculate time correlation functions in the Bak-Sneppen model [Phys. Rev. Lett. **71**, 4083 (1993)], a model showing self-organized criticality. For a random neighbor version of the model, analytical results are presented, while on a one-dimensional lattice we give numerical results. The power spectrum of these correlation functions shows $1/f$ behavior in both cases. [S1063-651X(96)08205-0]

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A large diversity of physical systems show $1/f$ noise [1]. The power spectra of time correlation functions of such systems show power law behavior $f^{-\beta}$ over many orders of magnitude with an exponent β in the range 0.6–1.6. A possible explanation for the wide occurrence of this phenomenon was put forward in a paper entitled ‘‘Self-organized criticality: An explanation of $1/f$ noise’’ [2]. In that paper, Bak, Tang, and Wiesenfeld [2] argue that many open nonlinear dynamical systems with a large number of degrees of freedom evolve to a state where they show critical behavior characterized by power law correlations both in space and time. Bak, Tang, and Wiesenfeld (BTW) illustrate their ideas using a simple model, the so-called sandpile model [3]. While this model shows many interesting properties, detailed investigations [4,5] showed that its power spectrum has f^{-2} behavior in any finite dimension. A mean-field calculation of the model did, however, show the expected $1/f$ behavior exactly [6].

Following the work of BTW a great variety of models (deterministic and stochastic, conservative and dissipative, etc.) have been introduced which show the phenomenon of self-organized criticality (SOC). A common feature of these models is the presence of a separation of time scales; the system is driven at a very slow rate until one of its elements reaches a threshold. This triggers a burst of activity (avalanche) which occurs on a very short time scale. When the avalanche is over, the system evolves again according to the slow drive until a next avalanche is triggered, and so on. The activity of the system in this way consists of a series of independent avalanches. A generic signature of SOC is the presence of a power law in the size (or duration) distribution of the avalanches. If one increases the external driving rate of the system this power law disappears. It was, however, shown by Hwa and Kardar [7] that if one increases the rate at which sand is dropped in the sandpile model, and one thus obtains the possibility of interacting avalanches, there appears a region in the power spectrum where the behavior is $1/f$.

The BTW-sandpile model is a stochastic and conservative model. Olami, Feder, and Christensen (OFC) [8] introduced a deterministic and dissipative model, related to spring-block models of earthquakes, which shows signatures of SOC, such as the occurrence of power law distributions for the sizes of the avalanches, with an exponent which depends on the degree of nonconservation in the model. In a subsequent study [9] it was shown that this model shows $1/f$ noise with an exponent β which also depends on the degree of nonconservation in the model. In a sense then, the OFC model fulfills, more than the sandpile model, the original requirements of the concept of SOC.

In the present paper we study the question of $1/f$ noise in the Bak-Sneppen model (BS) [10]. This model was introduced to describe the coevolution of species in the earth's ecology. Indeed the model shows many qualitative similarities with data from the real world, but fails on a quantitative level (see, e.g., [11]). In this paper we are only interested in the BS model as an interesting physical model and do not discuss its possible biological relevance. The occurrence of $1/f$ noise in the Bak-Sneppen model was already investigated numerically in [12]. Here we will introduce a somewhat different time correlation function, which has the advantage that it can be calculated exactly, at least in a mean-field theory.

In the BS model one has a system of N interacting species, each of which is represented by a real variable $x_i \in [0,1]$ (i: $1,\ldots,N$) which is a measure of the fitness of the species. Initially, all x_i are given a random value, taken from a uniform distribution on $[0,1]$. The dynamics of the model is defined as follows. First one looks for the site j where the fitness takes its lowest value. One then assigns a new random variable (taken again from the uniform distribution) x_j to species j. At the same time, the fitness of K other species is changed randomly. Several versions of the BS model can be defined, depending on the way in which these other species are chosen. In the lattice version of the model, the species are arranged on a lattice and the K species are taken as nearest neighbors. A random neighbor version, in which the K neighbors are chosen at random at each time step, was introduced in [13]. This version of the model has the advantage that several of its properties can be calculated exactly [14]. In this paper we will study both this random neighbor version (with $K=1$) and a one-dimensional version of the model in which we only modify the fitness of the neighbor to the right of the species with lowest fitness.

Analytical calculations and extensive simulations have shown that the BS model evolves to a state in which the probability distribution $p(x)$ that a species has a fitness x becomes a step function, which is zero for x less than some threshold value $x_c<1$, and which is $1/(1-x_c)$ for $x>x_c$. In the random neighbor model it is known that $x_c=1/(K+1)$ exactly. The exact value of x_c is not known for any lattice

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version of the model, but precise numerical estimates exist, especially in $d=1$, for the case in which both neighbors are updated [15,16]. For the case of the one-dimensional model in which one neighbor is updated, we know of no estimate for x_c in the literature. From our numerical results, we estimate $x_c=0.710\pm0.005$ for this case (details of our numerical work are described below).

Once the system has reached the equilibrium state, its dynamics is characterized by periods (identified with avalanches) in which at least one of the species has a fitness less than x_c, separated by periods in which all species have a fitness above threshold. The avalanches can be characterized either by their duration or by their total activity. Let us denote by $n(t)$ the number of species which are below threshold as a function of (discrete) time t. The total activity s of an avalanche lasting from $t=t_-$ to $t=t_+$ (so its total duration is $T=t_+-t_-+1$) is then given by

$$s=\sum_{t=t_-}^{t_+} n(t). \tag{1}$$

The distributions $P(T)$ of avalanche durations and $P(s)$ of avalanche sizes follow a power law

$$P(T)\sim T^{-\tau}, \quad P(s)\sim s^{-y}. \tag{2}$$

For the random neighbor model, it is known exactly that $\tau=3/2$ [14] while for the one-dimensional model (two neighbor updating) the most accurate numerical estimate is $\tau=1.073\pm0.003$ [15]. Our simulations of the one-dimensional one neighbor model lead to the estimate $\tau=1.08\pm0.01$, giving strong evidence that, as could be expected, both one-dimensional models are in the same universality class. We do not know of any existing estimates of the exponent y for the BS model.

It is of importance to remark that in the BS model as described so far there is no explicit time separation between a fast time scale for avalanches and a slow time scale for interavalanche periods. Such a separation is, however, *implicitly* present in the definition of the model since one assumes that one time step in the model is related to a step in ‘‘geological’’ time $t_g=\exp(x_{\min}/\tilde{T})$ (where $x_{\min}$ is the lowest value of x at a given time and T is a measure of mutation rate, see, e.g., [11]). When $1/\tilde{T}\gg1$, avalanches occur on time scales which are short compared to the time scale of the external drive which is set by the mutation rate. In this paper we will not further discuss the properties of the BS model on this geological time scale, since for that case no analytical results can be obtained and we want to concentrate on the properties of the BS model as a model in physics. If one wants to make a comparison with real world data, as is done, e.g., in [11], use of the time scale t_g is necessary.

In order to study spectral properties of the BS model it is necessary to introduce a dynamical correlation function $G_N(t)$. In [12] one uses the probability that a site which is active (i.e., has the lowest value of x) at time 0 will be active again at time t. In our calculations we use the autocorrelation of the total number of sites below threshold,

$$G_N(t)=\langle n(t_0)n(t_0+t)\rangle_{t_0}-\langle n(t_0)\rangle_{t_0}^2, \tag{3}$$

where the average is taken over time t_0 in the equilibrium state. The quantity $n(t)$ has the advantage that its time evolution can be rather easily studied in the random neighbor version of the BS model, as we discuss below. A rather similar quantity was also introduced in Flyvbjerg’s ‘‘pinball’’ approximation to the sandpile model [17]. If one compares with an equilibrium model, such as the Ising model, at criticality, our approach is like studying the autocorrelation function of the magnetization whereas the activity studied in [12] corresponds to measuring the probability that a spin that is flipped at time $t=0$ will be flipped again at time t.

According to the dynamical scaling hypothesis [18] one expects the Fourier transform $\hat{G}_N(\omega)$ of a correlation function such as (3) to scale as

$$\hat{G}_N(\omega)=\omega^{-\sigma}H(\omega N^z), \tag{4}$$

where H is a scaling function and z the dynamical exponent. Or equivalently, in real space

$$G_N(t)=N^{z(\sigma-1)}\tilde{H}(t/N^z). \tag{5}$$

We have calculated $G_N(t)$ analytically for the random neighbor version ($K=1$) of the BS model and numerically for the one-dimensional one neighbor version of the model. In both cases we find the presence of $1/f$ noise. We now turn to the details of these calculations, and we start with the analytical results.

In [14] a master equation approach to the random neighbor model was introduced. Let $P_n(t)$ be the probability that at time t, n species have a fitness which is below a certain value λ. In the end we will be most interested in the case when $\lambda=x_c$ but for the moment we look at the more general case. It is then rather easy to write down a master equation for $P_n(t)$,

$$P_n(t+1)=\sum_{m=0}^{N} M_{nm}P_m(t), \tag{6}$$

where the matrix elements M_{nm} can be written down in terms of λ and N [14]. For $t\to\infty, P_n(t)$ evolves to an equilibrium distribution P_n^*. The correlation function $G_N(t)$ can also be written down in terms of the matrix M. One has

$$G_N(t)=\lim_{t_0\to\infty}\sum_{m=0}^{N}\sum_{k=0}^{N} mkP_m(t_0)[M^tP(t_0)]_k - \left[\lim_{t_0\to\infty}\sum_{m=0}^{N} mP_m(t_0)\right]^2. \tag{7}$$

This expression in fact allows a (numerically) exact calculation of $G_N(t)$ in finite systems by simple iteration of the master equation (6). We have performed such calculations for $\lambda=x_c$ for systems with N up to 4000 and times t up to $2N$ (results are discussed below).

More interesting is the scaling limit in which $N\to\infty$ and $\lambda\to x_c$. In that limit it is possible to get a closed expression for the dynamic correlation function. It is therefore convenient to rewrite (7) as

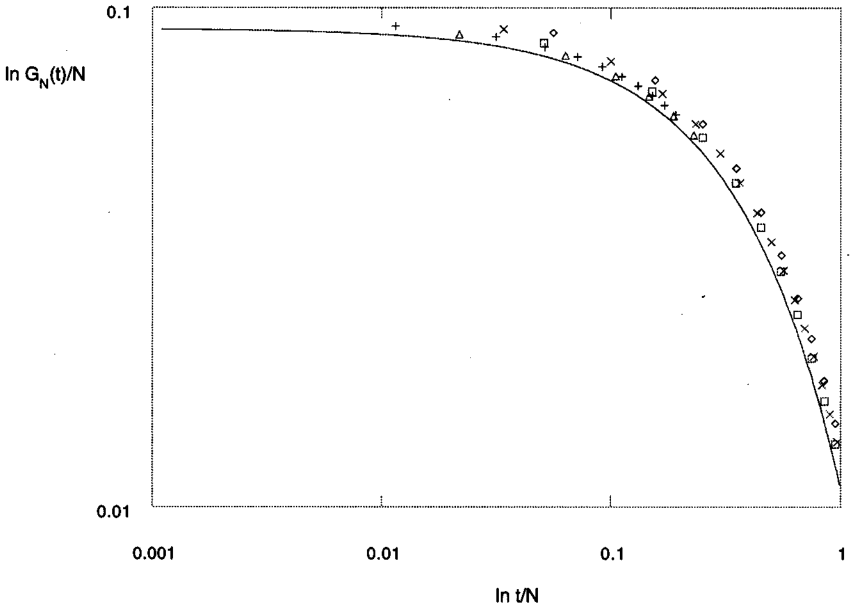

FIG. 1. The exact correlation function of the random neighbor Bak-Sneppen model. The figure shows the exact result (16) together with appropriately scaled finite system results obtained using (8). The results are for $N=250$ (◇), $N=500$ (×), $N=1000$ (□), $N=2000$ (+), and $N=4000$ (△).

$$G_N(t)=\sum_{m=0}^{N}\sum_{k=0}^{N} mkP_m^*Q_{mk}(t)-\left[\sum_{m=0}^{N} mP_m^*\right]^2, \qquad (8)$$

where $Q_{mk}(t)$ is the probability that in t time steps the number of species with fitness below λ changes from m to k. The authors of [14] assume that in the scaling limit P_n^* becomes a scaling function f of the variable $n/\sqrt{N}$,

$$P_n^*=\frac{1}{\sqrt{N}}f\left(\frac{n}{\sqrt{N}}\right). \qquad (9)$$

Inserting (9) into (6) and taking $t\rightarrow\infty$, $N\rightarrow\infty$, and $\lambda\rightarrow x_c$ then gives a differential equation from which f can be calculated [see Eq. (21) of [14]]. Using this result we immediately get the second term on the right-hand side of (8),

$$\left(\sum_{m=0}^{N} mP_m^*\right)^2=\frac{N}{2\pi}. \qquad (10)$$

What remains is a calculation of $Q_{mk}(t)$ in the scaling limit. We therefore assume that this probability scales as

$$Q_{mk}(t)=\frac{1}{\sqrt{N}}g\left(\frac{m}{\sqrt{N}},\frac{k}{\sqrt{N}},\frac{t}{N}\right). \qquad (11)$$

If we insert this assumption in (6) and take the scaling limit, we obtain a differential equation for g (with $x=k/\sqrt{N}, y=m/\sqrt{N}$, and $\tau=t/N$);

$$\frac{\partial g}{\partial \tau}=g+x\frac{\partial g}{\partial x}+\frac{1}{4}\frac{\partial^2 g}{\partial x^2}, \qquad (12)$$

which has to solved with the initial condition

$$g(x,y,\tau=0)=\delta(x-y) \qquad (13)$$

and reflecting boundary conditions in $x=0$.

The solution is

$$g(x,y,\tau)=h(x,y,\tau)+h(x,-y,\tau), \qquad (14)$$

where

$$h(x,y,\tau)=\sqrt{\frac{2}{\pi}}\left(\frac{1}{1-\exp(-2\tau)}\right)^{1/2}\exp(2y^2)\times\exp\left\{-\frac{2}{1-\exp(-2\tau)}[y^2+x^2-2xy\exp(-\tau)]\right\}. \qquad (15)$$

This result has to be used, together with (11), in the first term on the right-hand side of (8). Taking the scaling limit and using the expression of P_m^* from [14] we can rewrite this term as

$$N\frac{2\sqrt{2}}{\sqrt{\pi}}\left[\int_0^\infty dx\int_0^\infty dy\, xy\exp(-2y^2)g(x,y,\tau)\right].$$

Inserting our result for $g(x,y,\tau)$ and performing the integration then finally gives

$$G_N(t)=N\left\{\frac{1}{8\pi}[1-\exp(-2\tau)]^{3/2}[F(1,2,3/2,r_-(\tau))+F(1,2,3/2,r_+(\tau))-F(1,2,5/2,r_-(\tau))/3-F(1,2,5/2,r_+(\tau))/3]-\frac{1}{2\pi}\right\}, \qquad (16)$$

where $F(a,b,c,z)$ is the hypergeometric function and where

$$r_\pm(\tau)=\tfrac{1}{2}[1\pm\exp(-\tau)].$$

We thus see that the correlation function has indeed the scaling form (5) with $z=1$ and $\sigma=2$. In Fig. 1 we show our result (16) for $G_N(t)/N$ versus τ, together with the numerical results obtained from direct computation of (8) in finite sys-

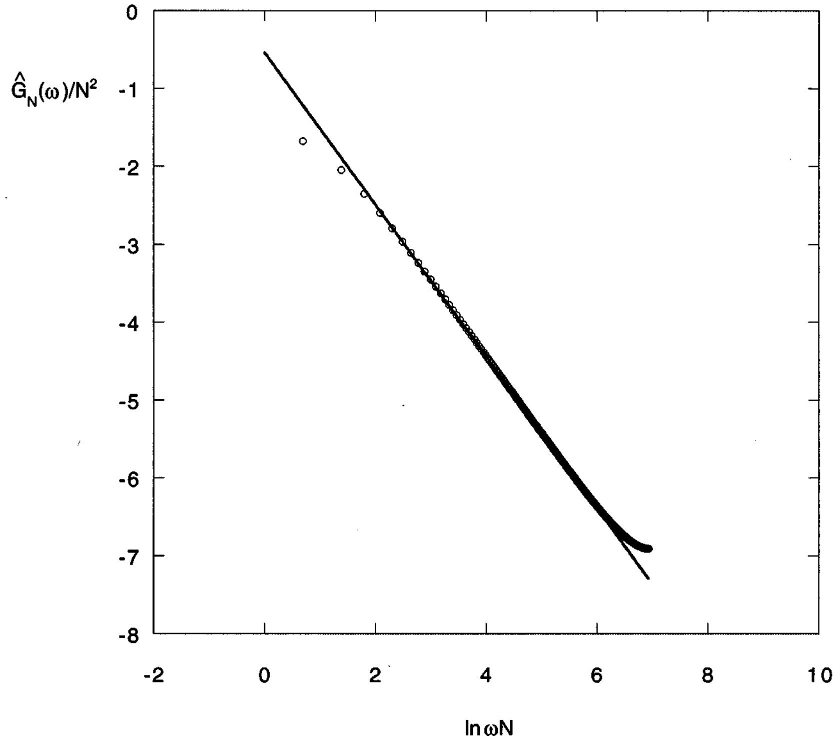

FIG. 2. Numerical Fourier transform (open circles) of the exact correlation function (16). The straight line represents a best fit through the linear part of the data and has a slope of -1.

tems. It is clear that the finite size data appproach the analytical results, thus lending support to the scaling assumptions we made.

To obtain the power spectrum we only have to Fourier transform (16). Unfortunately, we were not able to obtain an analytical expression for this transform. The result of a numerical transform using MATHEMATICA© is shown in Fig. 2. We show $\hat{G}_N(\omega)/N^2$ versus ωN, which are the natural scaling variables according to (4). The straight line shown has a slope -1. These results then show that over many orders of magnitude

$$\hat{G}_N(\omega)\sim\frac{N}{\omega} \tag{17}$$

so that indeed there is $1/f$ noise in the model.

It is interesting to remark here that the random neighbor versions of both the BTW-sandpile model [19] and the BS model [13] can be related to the critical branching process [20]. Within this approximation both models are thus in the same universality class. Since it is known that in a mean-field theory the sandpile model shows $1/f$ noise [6] it is not so surprising to find the same results for the BS model. Finally, we mention that the autocorrelation function of the activity introduced in [12] has a white noise power spectrum in the mean-field limit.

We now turn to a discussion of the one-dimensional one neighbor version of the BS model. Due to long range correlations which are present between subsequent species that

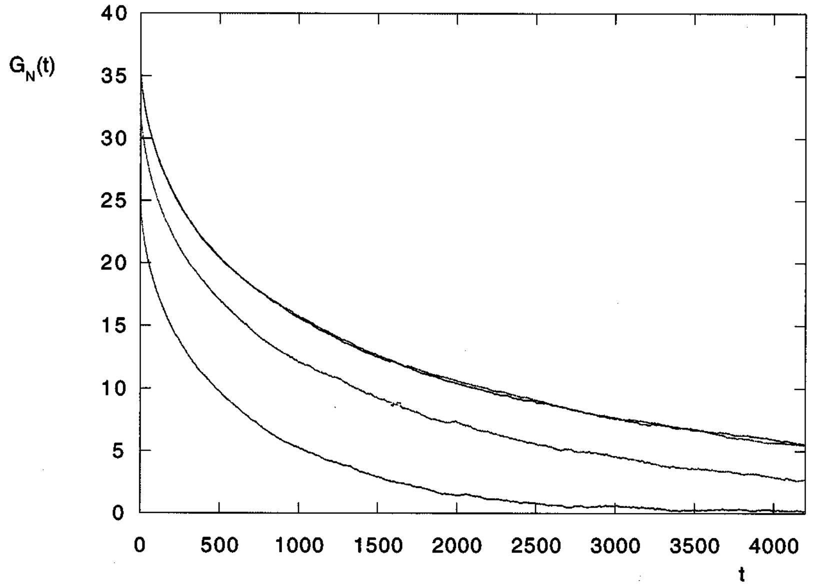

FIG. 3. Numerical results for the correlation function of the one-dimensional one neighbor Bak-Sneppen model. The different curves represent results for (bottom to top) $N=128{,}256{,}1024$, and 4096, respectively. The upper two curves almost completely coincide.

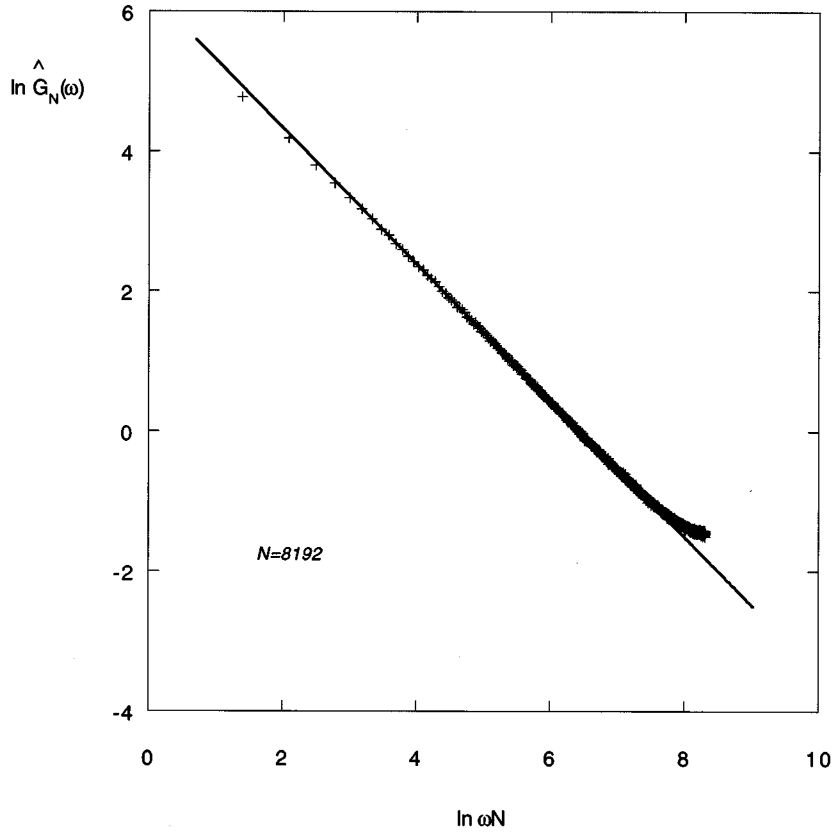

FIG. 4. Fourier transform (crosses) of the numerically calculated correlation function of the one-dimensional one neighbor Bak-Sneppen model. The results are for a system of $N=8192$ species. The straight line represents a best fit through the linear part of the data and has a slope of -0.972.

have lowest fitness [10] a master equation approach is no longer possible. So far, the only approach known for these lattice versions of the BS model is numerical. We have therefore performed extensive numerical calculations of the model on one-dimensional lattices with N up to 8192 and for time t up to 2^{32}. Using these data the values of x_c and τ for the one-dimensional one neighbor model mentioned above were obtained. Figure 3 shows numerical results for the correlation function $G_N(t)$ for various system sizes. Surprisingly, for large system sizes the correlation function seems to become independent of N, implying that z becomes 0. We do not fully understand this result, but it may be connected with similar behavior found for another exponent (η) in [16].

Figure 4 shows the power spectrum of the correlation function for the system with $N=8192$. As can be seen, the behavior is of the form $\omega^{-\beta}$ over many orders of magnitude. We estimate $\beta=0.97\pm0.05$. Thus, contrary to the sandpile model, the BS model has 1/f behavior also in a lattice version of the model. The exponent β is furthermore remarkably close to its mean-field value. In [12], an exponent $\beta=0.57$ was found for the power spectrum of the activity correlation function used in that reference.

In conclusion then, we have calculated the autocorrelation function of the number of species below threshold in the Bak-Sneppen model. For a random neighbor version of the model analytical results could be obtained, while in $d=1$ we obtained only numerical results. In both cases it was found numerically that the power spectrum shows 1/f behavior.

We are currently investigating the correlation function in random neighbor versions of other models of SOC, in the hope of obtaining analytical results, and investigating the presence of 1/f noise also in those models.

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